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Givental Integral Representation for Classical Groups [1]

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Abstract

We propose integral representations for wave functions of B_n, C_n, and D_n open Toda chains at zero eigenvalues of the Hamiltonian operators thus generalizing Givental representation for A_n. We also construct Baxter Q-operators for closed Toda chains corresponding to Lie algebras B_∞, C_∞, D_∞, affine Lie algebras $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and twisted affine Lie algebras $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$. Our approach is based on a generalization of the connection between Baxter Q-operator for $A_n^{(1)}$ closed Toda chain and Givental representation for the wave function of A_n open Toda chain uncovered previously.

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1 Introduction

A remarkable integral representation for the common eigenfunctions of A_n open Toda chain Hamiltonian operators was proposed in [Gi] (see also [JK]). This representation is based on a flat degeneration of A_n flag manifolds to a Gorenstein toric Fano variety (see [L],[Ba], [BCFKS] for details). This results in a purely combinatorial description of the integrand in the integral representation [Gi]. An important application of the Givental integral representation so far was an explicit construction of the mirror dual of A_n flag manifolds.

Later it turns out that the representation introduced in [Gi] is also interesting from another points of view. Thus it was shown in [GKLO] that this integral representation has natural iterative structure allowing the connection of A_{n-1} and A_n wave functions by a simple integral transformation. It was demonstrated that thus defined integral transformation is given by a degenerate version of the Baxter Q-operator realizing quantum Bäcklund transformations in closed Toda chain [PG]. Let us note that the torification of flag manifolds leads to a distinguished set of coordinates on its open parts. A group theory construction of these coordinates is also connected with a degenerate Q-operator and was clarified in [GKLO].

Up to now the Givental integral representation was only generalized [BCFKS] to the case of degenerate A_n open Toda chains [STS] leading to a construction of the mirror duals to partial flag manifolds G/P for $G = SL(n+1, \mathbb{C})$ and P being a parabolic subgroup. A natural approach to a generalization of the integral representation to other Lie algebras could be based on the relation with Baxter Q-operator. However no generalization of the Baxter Q-operator to other Lie algebras was known. In this note we solve both these problems simultaneously for all classical series of Lie algebras. We propose a generalization of the Givental integral representation to other classical series B_n, C_n, D_n and construct Q-operators for affine Lie algebras $A_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and infinite Lie algebras B_∞, C_∞ and D_∞. We also generalize the connection between Q-operators and integral representations of wave functions to all classical series.

Let us stress that there is an important difference in the construction of the integral representations between A_n and other classical series. The kernel of the integral operator providing recursive construction of the integral representation for A_n has a simple form of the exponent of the sum of exponents in the natural coordinates. For other classical groups recursive operators of the same type exist but they relate Toda chain wave functions for different classical series (e.g. C_n and D_n). Integral operators connecting Toda wave functions in the same series (e.g. D_n and D_{n-1}) are given by compositions of the elementary integral operators.

In this note we restrict ourselves by explicit constructions of the integral representations of eigenfunctions of the quadratic open Toda chain Hamiltonian operators at zero eigenvalues. The general case of all Hamiltonians and non-zero eigenvalues will be published elsewhere. Also we leave for another occasion the elucidation of a group theory interpretation of the obtained results.

The plan of this paper is as follows. In Section 2 we summarize the results of [GKLO]. In Section 3 we construct kernels of the elementary integral operators intertwining Hamiltonian operators of open Toda chains for (in general different) classical series of finite Lie algebras.

In Section 4 using the results from Section 3 we give explicit integral representations for the wave functions of B_n, C_n and D_n open Toda chains. In Section 5 we describe a generalization of Givental diagrams to other classical series and remark on the connection with toric degeneration of B_n, C_n and D_n flag manifolds. In Section 6 we construct elementary integral operators intertwining Hamiltonian operators of closed Toda chains for (in general different) classical series of affine Lie algebras. In Section 7 we construct integral kernels for Baxter Q-operators for all classical series of (twisted) affine Lie algebras as appropriate compositions of the elementary intertwining operators. In Section 8 we construct Q operators for B_∞, C_∞ and D_∞ Toda chains. We conclude in Section 9 with a short discussion of the results presented in this note.

We were informed by E. Sklyanin that he also has some progress in the construction of Baxter Q-operators for Toda theories.

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2 Recursive structure of Givental representation

In this section we recall a recursive construction of the Givental integral representation discussed in [GKLO].

The solution of a quantum integrable system starts with the finding of the full set of common eigenfunctions of the quantum Hamiltonian operators of A_n Toda chain. Note that the difference between wave functions for $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ manifests only at non-zero eigenvalues of the element of the center of $U\mathfrak{gl}_n$ linear over the generators. In the following we will consider only the wave functions corresponding to zero eigenvalues of all elements of the center. Thus we will always consider $\mathfrak{gl}_n$ Toda chain instead of A_n Toda chain. The quadratic quantum Hamiltonian of $\mathfrak{gl}_n$ open Toda chain is given by

$$H^{\mathfrak{gl}_n}(x) = -\frac{\hbar^2}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n-1} g_i e^{x_{i+1}-x_i}\,. \tag{2.1}$$

In [Gi] the following remarkable representation for a common eigenfunction of quantum Hamiltonians of $\mathfrak{gl}_n$ open Toda chain was proposed

$$\Psi(x_1,\dots,x_n) = \int_\Gamma e^{\frac{1}{\hbar}\mathcal{F}_n(x)} \bigwedge_{k=1}^{n-1}\bigwedge_{i=1}^{k} dx_{k,i}, \tag{2.2}$$

where $x_{n,i} := x_i$, the function $\mathcal{F}_n(x)$ is given by

$$\mathcal{F}_n(x) = \sum_{k=1}^{n-1}\sum_{i=1}^{k}\Big(e^{x_{k,i}-x_{k+1,i}} + g_i e^{x_{k+1,i+1}-x_{k,i}}\Big), \tag{2.3}$$

and the cycle Γ is a middle dimensional submanifold in the $n(n-1)/2$- dimensional complex torus with coordinates $\{\exp x_{k,i},\ i = 1,\ldots,k;\ k = 1,\ldots,n-1\}$ such that the integral converges. The eigenfunction (2.2) solves the equation

$$H^{\mathfrak{gl}_n}(x)\,\Psi(x_1,\cdots,x_n) = 0. \tag{2.4}$$

In the following we put $\hbar = 1$ for convenience.

The derivation of the integral representation (2.2) using the recursion over the rank n of the Lie algebra $\mathfrak{gl}_n$ was given in [GKLO]. The integral representation for the wave function can be represented in the following form

$$\Psi(x_1,\ldots,x_n) \;=\; \int \bigwedge_{k=1}^{n-1}\bigwedge_{i=1}^{k} dx_{k,i}\,\prod_{k=1}^{n-1} Q_{k+1,\,k}(x_{k+1,1},\ldots,x_{k+1,k+1};\,x_{k,1},\ldots,x_{k,k}), \tag{2.5}$$

with the integral kernel

$$Q_{k+1,k}(x_{k+1,i};\,x_{k,i}) = \exp\Big\{\,\sum_{i=1}^{k}\big(e^{x_{k,i}-x_{k+1,i}} + g_i e^{x_{k+1,i+1}-x_{k,i}}\big)\,\Big\}. \tag{2.6}$$

Here we have $x_i := x_{n,i}$. The following differential equation for the kernel holds

$$H^{\mathfrak{gl}_{k+1}}(x_{k+1,i})Q_{k+1,k}(x_{k+1,i},\,x_{k,i}) = Q_{k+1,k}(x_{k+1,i},\,x_{k,i})\,H^{\mathfrak{gl}_k}(x_{k,i}), \tag{2.7}$$

where

$$H^{\mathfrak{gl}_k}(x_i) = -\frac{1}{2}\sum_{i=1}^{k}\frac{\partial^2}{\partial x_{k,i}^2} + \sum_{i=1}^{k-1} g_i e^{x_{k,i+1}-x_{k,i}}\,. \tag{2.8}$$

Here and in the following we assume that in the relations similar to (2.7) the Hamiltonian operator on l.h.s. acts on the right and the Hamiltonian on r.h.s. acts on the left. Thus the integral operator with the kernel $Q_{k+1,k}$ intertwines Hamiltonian operators for $\mathfrak{gl}_{k+1}$ and $\mathfrak{gl}_k$ open Toda chains.

The integral operator defined by the kernel (2.6) is closely related with a Baxter Q-operator realizing Bäcklund transformations in a closed Toda chain corresponding to affine Lie algebra $\widehat{\mathfrak{gl}}_n$. Baxter Q-operator for zero spectral parameter can be written in the integral form with the kernel [PG]

$$Q^{\widehat{\mathfrak{gl}}_n}(x_i,y_i) = \exp\Big\{\,\sum_{i=1}^{n}\big(e^{x_i-y_i} + g_i e^{y_{i+1}-x_i}\big)\,\Big\}, \tag{2.9}$$

where $x_{i+n} = x_i$ and $y_{i+n} = y_i$. This operator commutes with the Hamiltonian operators of the closed Toda chain. Thus for example for the quadratic Hamiltonian we have

$$\mathcal{H}^{\widehat{\mathfrak{gl}}_n}(x_i)Q^{\widehat{\mathfrak{gl}}_n}(x_i,\,y_i) = Q^{\widehat{\mathfrak{gl}}_n}(x_i,\,y_i)\mathcal{H}^{\widehat{\mathfrak{gl}}_n}(y_i), \tag{2.10}$$

where

$$\mathcal{H}^{\widehat{\mathfrak{gl}}_n} = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n} g_i e^{x_{i+1}-x_i}\,. \tag{2.11}$$

Here we impose the conditions $x_{i+n} = x_i$. The recursive operator (2.6) can be obtained from Baxter operator (2.9) in the limit $g_n \to 0$, $x_n \to -\infty$.

The main objective of this note is to generalize the representation (2.2), (2.3) to other classical series B_n, C_n and D_n of finite Lie groups. Before we present the integral representations for B_n, C_n and D_n let us comment on the main subtlety in their constructions. As it was explained in [GKLO] the variables $x_{k,i}$ in the integral representation for A_n have a clear meaning of the linear coordinates on Cartan subalgebras of the intermediate Lie algebras entering recursion $A_n \to A_{n-1} \to \cdots \to A_1$. This is a consequence of the identity

$$(\dim(\mathfrak{gl}_n) - \mathrm{rk}\,(\mathfrak{gl}_n)) - (\dim(\mathfrak{gl}_{n-1}) - \mathrm{rk}\,(\mathfrak{gl}_{n-1})) = 2\,\mathrm{rk}\,\mathfrak{gl}_{n-1}. \tag{2.12}$$

However for other classical series there is no such simple relation. In general one finds more integration variables in the integral representation then those arising as linear coordinates on intermediate Cartan subalgebras. It turns out that the elementary integral operators intertwine Hamiltonians corresponding to Toda chains for *different* Lie algebras. The recursive operators are then constructed as appropriate compositions of the elementary intertwining operators.

3 Elementary intertwiners for open Toda chains

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra, $n = \dim\mathfrak{h}$ be the rank of $\mathfrak{g}$, $R \subset \mathfrak{h}^*$ be the root system, W be the Weyl group. Let us fix a decomposition $R = R_+ \cup R_-$ of the roots on positive and negative roots. Let $\alpha_1, \ldots, \alpha_n$ be the bases of simple roots. Let $(,)$ be a W-invariant bilinear symmetric form on $\mathfrak{h}^*$ normalized so that $(\alpha, \alpha) = 2$ for a long root. This form provides an identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ and thus can be considered as a bilinear form on $\mathfrak{h}$. Choose an orthonormal basis $e = \{e_1, \ldots, e_n\}$ in $\mathfrak{h}$. Then for any $x \in \mathfrak{h}$ one has a decomposition $x = \sum_{i=1}^n x_i e_i$. One associates with these data an open Toda chain with a quadratic Hamiltonian

$$H^R(x_i) = -\frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^n g_i e^{\alpha_i(x)}\,. \tag{3.1}$$

For the standard facts on Toda theories corresponding to arbitrary root systems see e.g. [RSTS].

We start with explicit expressions for elementary intertwining operators for open Toda chains. The necessary facts on the root systems (including non-reduced ones) can be found in [He].

3.1 $BC \leftrightarrow B$

Let $e = \{e_1, \ldots, e_n\}$ be an orthonormal basis in $\mathbb{R}^n$. Non-reduced root system of type BC_n can be defined as

$$\alpha_0 = 2e_1, \qquad \alpha_1 = e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad 1 \le i \le n-1 \tag{3.2}$$

and the corresponding Dynkin diagram is

$$\frac{\alpha_0}{\alpha_1} \Longleftrightarrow \alpha_2 \longleftarrow \ldots \longleftarrow \alpha_n \tag{3.3}$$

where the first vertex from the left is a doubled vertex corresponding to a reduced $\alpha_1 = e_1$ and non-reduced $\alpha_0 = 2e_1 = 2\alpha_1$ roots.

Quadratic Hamiltonian operator of the corresponding open Toda chain is given by

$$H^{BC_n}(x_i) = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{g_1}{2}\Big(e^{x_1} + g_1 e^{2x_1}\Big) + \sum_{i=1}^{n-1} g_{i+1} e^{x_{i+1}-x_i}. \tag{3.4}$$

Let us stress that the same open Toda chain can be considered as a most general form of C_n open Toda chain (see e.g. [RSTS], Remark p.61). However in the following we will use the term BC_n open Toda chain to distinguish it from a more standard C_n open Toda chain that will be consider below.

The root system of type B_n can be defined as

$$\alpha_1 = e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad 1 \leq i \leq n-1 \tag{3.5}$$

and the corresponding Dynkin diagram is

$$\alpha_1 \Longleftarrow \alpha_2 \longleftarrow \ldots \longleftarrow \alpha_n\,. \tag{3.6}$$

Quadratic Hamiltonian operator of the corresponding open Toda chain is given by

$$H^{B_n}(x_i) = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + g_1 e^{x_1} + \sum_{i=1}^{n-1} g_{i+1} e^{x_{i+1}-x_i}. \tag{3.7}$$

An elementary operator intertwining open Toda chain Hamiltonians for BC_n and B_{n-1} can be written in the integral form with the kernel

$$Q_{BC_n}^{B_{n-1}}(z_1, \ldots, z_n;\, x_1, \ldots, x_{n-1}) = \exp\Big\{\, g_1 e^{z_1} + \sum_{i=1}^{n-1}(e^{x_i - z_i} + g_{i+1} e^{z_{i+1}-x_i})\,\Big\}, \tag{3.8}$$

satisfying the following relation

$$H^{BC_n}(z)\, Q_{BC_n}^{B_{n-1}}(z,\, x) = Q_{BC_n}^{B_{n-1}}(z,\, x)\, H^{B_{n-1}}(x). \tag{3.9}$$

Similarly an elementary operator intertwining B_n and BC_n Hamiltonians has an integral kernel

$$\begin{gathered} Q_{B_n}^{BC_n}(x_1, \ldots, x_n;\, z_1, \ldots, z_n) = \\ \exp\Big\{\, g_1 e^{z_1} + \sum_{i=1}^{n-1}\Big(e^{x_i - z_i} + g_{i+1} e^{z_{i+1}-x_i}\Big) + e^{x_n - z_n}\,\Big\}. \end{gathered} \tag{3.10}$$

3.2 $C \leftrightarrow D$

The root system of type C_n can be defined as

$$\alpha_i = e_{i+1} - e_i, \qquad \alpha_n = 2e_n, \qquad 1 \le i \le n-1, \tag{3.11}$$

and the corresponding Dynkin diagram is

$$\alpha_1 \longleftarrow \ \ldots \longleftarrow \alpha_{n-1} \Longleftarrow \alpha_n\,. \tag{3.12}$$

Quadratic Hamiltonian operator of the corresponding open Toda chain is given by

$$H^{C_n}(x_i) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n-1} g_i e^{x_{i+1}-x_i} + 2g_n e^{-2x_n}. \tag{3.13}$$

The root system of type D_n is

$$\alpha_i = e_{i+1} - e_i, \qquad \alpha_n = -e_{n-1} - e_n, \qquad 1 \le i < n, \tag{3.14}$$

and the corresponding Dynkin diagram is

$$\begin{array}{ccccccc} \alpha_1 & \longrightarrow & \ldots & \longrightarrow & \alpha_{n-2} & \longrightarrow & \alpha_{n-1} \\ & & & & \Big\downarrow & & \\ & & & & \alpha_n & & \end{array} \tag{3.15}$$

Quadratic Hamiltonian operator of the D_n open Toda chain is given by

$$H^{D_n}(x_i) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n-1} g_i e^{x_{i+1}-x_i} + g_{n-1}g_n e^{-x_n-x_{n-1}}. \tag{3.16}$$

An integral operator intertwining C_n and D_n Hamiltonians has the kernel

$$\begin{gathered} Q_{C_n}^{D_n}(z_1,\ldots,z_n;x_1,\ldots,x_n) = \\ \exp\Big\{\sum_{i=1}^{n-1}\Big(e^{x_i-z_i} + g_i e^{z_{i+1}-x_i}\Big) + e^{x_n-z_n} + g_n e^{-x_n-z_n}\Big\}. \end{gathered} \tag{3.17}$$

Similarly an integral operator with the kernel

$$\begin{gathered} Q_{D_n}^{C_{n-1}}(x_1,\ldots,x_n;\, z_1,\ldots,z_{n-1}) = \\ = \exp\Big\{\sum_{i=1}^{n-1}\Big(e^{z_i-x_i} + g_i e^{x_{i+1}-z_i}\Big) + g_n e^{-x_n-z_{n-1}}\Big\}, \end{gathered} \tag{3.18}$$

intertwines the following D_n and C_{n-1} quadratic Hamiltonians

$$H^{D_n}(x_i) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n-1} g_i e^{x_{i+1}-x_i} + g_n e^{-x_n-x_{n-1}}, \tag{3.19}$$

$$H^{C_{n-1}}(z_i) = -\frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial^2}{\partial z_i^2} + \sum_{i=1}^{n-2} g_i e^{z_{i+1}-z_i} + 2g_{n-1}g_n e^{-2z_{n-1}}. \tag{3.20}$$

4 Givental representation for wave functions

In the previous section explicit expressions for the kernels of elementary intertwining operators were presented. Now integral representations for eigenfunctions of open Toda chain Hamiltonians are given by a quite straightforward generalization of A_n case. Below we provide integral representations for all classical series. For simplicity we put $g_i = 1$ below.

4.1 B_n

The eigenfunction for B_n open Toda chain is given by

$$\Psi^{B_n}(x_1,\dots,x_n) \;=\; \int \bigwedge_{k=1}^{n-1}\bigwedge_{i=1}^{k} dx_{k,i}\, \prod_{k=1}^{n-1} Q_{B_{k+1}}^{B_k}(x_{k+1,1},\dots,x_{k+1,k+1};\; x_{k,1},\dots,x_{k,k}), \tag{4.1}$$

where $x_i := x_{n,i}$ and the kernels $Q_{B_{k+1}}^{B_k}$ of the integral operators are given by the convolutions of the kernels $Q_{B_{k+1}}^{BC_{k+1}}$ and $Q_{BC_{k+1}}^{B_k}$

$$\begin{aligned} Q_{B_{k+1}}^{B_k}(x_{k+1,i};\; x_{k,j}) = \int \bigwedge_{i=1}^{k} dz_{k,i}\, Q_{B_{k+1}}^{BC_{k+1}}(x_{k+1,1},\dots,x_{k+1,k+1}; z_{k+1,1},\dots,z_{k+1,k+1}) \times \\ \times Q_{BC_{k+1}}^{B_k}(z_{k+1,1},\dots,z_{k+1,k+1}; x_{k,1},\dots,x_{k,k}). \end{aligned} \tag{4.2}$$

Notice that the wave function is given by the integral over a contour of the real dimension equal to a complex dimension of the flag manifold $X = G/B$, where $G = SO(2n+1,\mathbb{C})$ and B is a Borel subgroup

$$\sum_{k=1}^{n}(2k-1) = n^2 = |R_+|. \tag{4.3}$$

4.2 C_n

The eigenfunction for C_n open Toda chain is given by

$$\Psi^{C_n}(z_1,\dots,z_n) \;=\; \int \bigwedge_{k=1}^{n-1}\bigwedge_{i=1}^{k} dz_{k,i}\, \prod_{k=1}^{n-1} Q_{C_{k+1}}^{C_k}(z_{k+1,1},\dots,z_{k+1,k+1};\; z_{k,1},\dots,z_{k,k}), \tag{4.4}$$

where $z_i := z_{n,i}$ and the kernels $Q_{C_{k+1}}^{C_k}$ of the integral operators are given by the convolutions of the kernels $Q_{C_{k+1}}^{D_{k+1}}$ and $Q_{D_{k+1}}^{C_k}$

$$\begin{aligned} Q_{C_{k+1}}^{C_k}(z_{k+1,i};\; z_{k,j}) = \int \bigwedge_{i=1}^{k} dx_{k,i}\, Q_{C_{k+1}}^{D_{k+1}}(z_{k+1,1},\dots,z_{k+1,k+1}; x_{k+1,1},\dots,x_{k+1,k+1}) \times \\ \times Q_{D_{k+1}}^{C_k}(x_{k+1,1},\dots,x_{k+1,k+1}; z_{k,1},\dots,z_{k,k}). \end{aligned} \tag{4.5}$$

Thus the wave function is given by the integral over a contour of the real dimension equal to a complex dimension of the flag manifold $X = G/B$, where $G = Sp(n, \mathbb{C})$ and B is a Borel subgroup

$$\sum_{k=1}^{n}(2k-1) = n^2 = |R_+|. \tag{4.6}$$

4.3 D_n

The eigenfunction for D_n open Toda chain is given by

$$\Psi^{D_n}(x_1,\dots,x_n) \;=\; \int \bigwedge_{k=1}^{n-1}\bigwedge_{i=1}^{k} dx_{k,i}\,\prod_{k=1}^{n-1} Q_{D_{k+1}}^{D_k}(x_{k+1,1},\dots,x_{k+1,k+1};\,x_{k,1},\dots,x_{k,k}), \tag{4.7}$$

where $x_i := x_{n,i}$ and the kernels $Q_{D_{k+1}}^{D_k}$ of the integral operators are given by the convolutions of the kernels $Q_{D_{k+1}}^{C_k}$ and $Q_{C_k}^{D_k}$

$$Q_{D_{k+1}}^{D_k}(x_{k+1,i};\,x_{k,j}) = \int \bigwedge_{i=1}^{k} dz_{k,i}\,Q_{D_{k+1}}^{C_k}(x_{k+1,1},\dots,x_{k+1,k+1};z_{k,1},\dots,z_{k,k})\times \tag{4.8}$$
$$\times Q_{C_k}^{D_k}(z_{k,1},\dots,z_{k,k};x_{k,1},\dots,x_{k,k}).$$

Thus the wave function is given by the integral over a contour of the real dimension equal to a complex dimension of the flag manifold $X = G/B$, where $G = SO(2n, \mathbb{C})$ and B is a Borel subgroup

$$\sum_{k=1}^{n-1} 2k = n(n-1) = |R_+|. \tag{4.9}$$

5 Givental diagrams and Toric degenerations

In this section we describe combinatorial structure entering the integral representations presented above. This structure reflects flat toric degenerations of the corresponding flag manifolds (see [BCFKS] for A_n and [B] for a general approach to mirror symmetry via degeneration). The combinatorial structure of the integrand readily encoded into the (generalized) Givental diagrams. Let us note that the diagrams for B_n, C_n and D_n can be obtained from those for A_n by a factorization. This factorization is closely related with a particular realization [DS] of classical series of Lie algebras as fixed point subalgebras of the involutions acting on an algebra $\mathfrak{gl}_N$ for some N.

5.1 A_n Diagram

Givental diagram [Gi] for A_n has the following form

$$
\begin{tikzcd}
x_{n+1,n+1} & & & & \\
x_{n,n} \arrow[u, "b_{n,n}"'] & x_{n+1,n} \arrow[l] & & & \\
\vdots \arrow[u, "b_{n-1,n-1}"'] & \vdots \arrow[u, "b_{n,n-1}"'] & \ddots & & \\
x_{2,2} \arrow[u, "b_{2,2}"'] & x_{3,2} \arrow[l, "a_{2,2}"'] \arrow[u, "b_{3,2}"'] & \dots \arrow[l, "a_{3,2}"'] & x_{n+1,2} \arrow[l, "a_{n,2}"'] & \\
x_{1,1} \arrow[u, "b_{1,1}"'] & x_{2,1} \arrow[l, "a_{1,1}"'] \arrow[u, "b_{2,1}"'] & \dots \arrow[l, "a_{2,1}"'] & x_{n,1} \arrow[l, "a_{n-1,1}"'] \arrow[u, "b_{n,1}"'] & x_{n+1,1} \arrow[l, "a_{n,1}"']
\end{tikzcd}
\tag{5.1}
$$

We assign variables $x_{k,i}$ to the vertexes (k,i) and functions e^{y-x} to the arrows $(x \longrightarrow y)$ of the diagram (5.1). The potential function $\mathcal{F}(x_{k,i})$ (see (2.3)) is given by the sum of the functions assigned to all arrows.

Note that the variables $\{x_{k,i}\}$ naturally parametrize an open part U of the flag manifold $X = SL(n+1,\mathbb{C})/B$. The non-compact manifold U has a natural action of the torus and can be compactified to a (singular) toric variety. The set of the monomial relations defining this compactifiaction can be described as follows. Let us introduce the new variables

$$
a_{k,i} = e^{x_{k,i}-x_{k+1,i}}, \quad b_{k,i} = e^{x_{k+1,i+1}-x_{k,i}}, \qquad 1 \le k \le n,\ 1 \le i \le k
$$

assigned to the arrows of the diagram (5.1). Then the following defining relations hold

$$
\begin{gathered}
a_{k,i} \cdot b_{k,i} \;=\; b_{k+1,i} \cdot a_{k+1,i+1}, \qquad 1 \le k < n,\ 1 \le i \le k \\
a_{n,i} \cdot b_{n,i} = e^{x_{n,i+1}-x_{n,i}}
\end{gathered}
\tag{5.2}
$$

The defining relations of the toric embedding are given by the monomial relations for the variables associated with the paths on the diagram. They are given by a simple generalization of the relations (5.2) (see [BCFKS] for details).

5.2 B_n Diagram

Diagram for B_n has the following form ($n = 3$)

$$
\begin{tikzcd}
& & & {} \arrow[d, "b_{31}"'] & & & \\
& & {} \arrow[r, "a_{31}"] \arrow[d, "b_{21}"'] & z_{31} \arrow[r, "c_{31}"] \arrow[d, "d_{21}"'] & x_{31} \arrow[d, "b_{32}"'] & & \\
& {} \arrow[r, "a_{21}"] \arrow[d, "b_{11}"'] & z_{21} \arrow[r, "c_{21}"] \arrow[d, "d_{11}"'] & x_{21} \arrow[r, "a_{32}"] \arrow[d, "b_{22}"'] & z_{32} \arrow[r, "c_{32}"] \arrow[d, "d_{22}"'] & x_{32} \arrow[d, "b_{33}"'] & \\
{} \arrow[r, "a_{11}"] & z_{11} \arrow[r, "c_{11}"] & x_{11} \arrow[r, "a_{22}"] & z_{22} \arrow[r, "c_{22}"] & x_{22} \arrow[r, "a_{33}"] & z_{33} \arrow[r, "c_{33}"] & x_{33}
\end{tikzcd}
\tag{5.3}
$$

Here we use the same rules for assigning variables to the arrows of the diagram as in A_n case. In addition we assign functions e^x to the arrows ($\longrightarrow x$).

Note that the diagram for B_n can be obtained by factorization of the diagram (5.1) for A_{2n} by the following involution

$$\iota: \quad X \longmapsto w_0^{-1} X^T w_0, \tag{5.4}$$

where w_0 is the longest element of A_{2n} Weyl group $\mathfrak{W}(A_{2n})$ isomorphic to a symmetric group $\mathfrak{S}_{2n+1}$ and X^T denotes the standard transposition. Correspondingly the diagram for B_n can be obtained from A_{2n} diagram by the quotient

$$w_0: \qquad x_{k,i} \longleftrightarrow -x_{k,k+1-i}. \tag{5.5}$$

An analog of the monomial relations (5.2) is as follows. Associate to the arrows of Givental diagram parameters

$$\begin{gathered} a_{k,i} = e^{z_{k,i} - x_{k-1,i-1}}, \;\; b_{k,i} = e^{z_{k,i} - x_{k,i-1}}, \;\; c_{k,i} = e^{z_{k,i} - x_{k,i}}, \;\; d_{l,j} = e^{x_{l,j} - z_{l+1,j}} \\ 1 \leq k \leq n, \; 1 \leq i \leq k, \quad 1 \leq l \leq n-1, \; 1 \leq j \leq l. \end{gathered} \tag{5.6}$$

Then the following relations hold:

$$\begin{array}{rcll} a_{k,1} & = & b_{k,1}, & 1 \leq k \leq n, \\ d_{k,i} \cdot a_{k+1,i+1} & = & c_{k+1,i} \cdot b_{k+1,i+1}, & 1 \leq k < n-1, \; 1 \leq i \leq k \\ b_{k,i} \cdot c_{k,i} & = & a_{k+1,i} \cdot d_{k,i}, & 1 \leq k < n-1, \; 1 \leq i \leq k \\ b_{n,i} \cdot c_{n,i} & = & e^{x_{n,i} - x_{n,i-1}} & \end{array} \tag{5.7}$$

5.3 C_n Diagram

Diagram for C_n has the following form $(n=3)$

$$
\begin{tikzcd}
 & & z_{33} \arrow[d, equal] & & & \\
 & z_{22} \arrow[r, equal] \arrow[d, equal] & x_{33} & z_{33} \arrow[l, "c_{33}"'] & & \\
z_{11} \arrow[r, equal] \arrow[d, equal] & x_{22} & z_{22} \arrow[l, "c_{22}"'] \arrow[u, "d_{22}"] & x_{32} \arrow[l, "a_{32}"'] \arrow[u, "b_{32}"] & z_{32} \arrow[l, "c_{32}"'] & \\
x_{11} & z_{11} \arrow[l, "c_{11}"'] \arrow[u, "d_{11}"] & x_{21} \arrow[l, "a_{21}"'] \arrow[u, "b_{21}"] & z_{21} \arrow[l, "c_{21}"'] \arrow[u, "d_{21}"] & x_{31} \arrow[l, "a_{31}"'] \arrow[u, "b_{31}"] & z_{31} \arrow[l, "c_{31}"']
\end{tikzcd}
\tag{5.8}
$$

where one assigns functions e^{-z-x} to double arrows ($x = z$).

The Lie algebra C_n can be realized as a fixed point subalgebra of A_{2n-1} using the involution

$$
\iota : \quad X \longmapsto w_0^{-1} X^T w_0, \tag{5.9}
$$

where w_0 is the longest element of Weyl group $\mathfrak{W}(A_{2n-1}) = \mathfrak{S}_{2n}$ and X^T denotes the standard transposition. Correspondingly the diagram for C_n can be obtained from A_{2n-1} diagram by the quotient

$$
w_0 : \qquad x_{k,i} \longleftrightarrow -x_{k,k+1-i}. \tag{5.10}
$$

Note that diagram for C_n can be also obtained by erasing the last row of vertexes and arrows on the right slope from the diagram for D_{n+1} (see (5.13) below).

An analog of the monomial relations (5.2) is as follows. Let us introduce the variables

$$
\begin{aligned}
a_{l,j} &= e^{z_{l-1,j}-x_{l,j}}, & & 1 < l \le n,\ 1 \le j \le l,\ l \ne j, \\
b_{k,i} &= e^{z_{k,i+1}-x_{k,i}}, & & 1 \le k \le n,\ 1 \le i \le k,\ k \ne i, \\
c_{k,i} &= e^{x_{k,i}-z_{k,i}}, & & 1 \le k \le n,\ 1 \le i \le k, \\
d_{m,j} &= e^{x_{m+1,j+1}-z_{m,j}}, & & 1 \le m < n,\ 1 \le j \le m
\end{aligned}
\tag{5.11}
$$

where one assign the variables b_{11}, a_{22}, b_{22}, a_{33}, and b_{33} to the left slope of the diagram (5.8). Then the following relations hold

$$
\begin{gathered}
c_{k,i} \cdot b_{k,i} \;=\; d_{k,i} \cdot a_{k+1,i+1}, \\
a_{k,i} \cdot d_{k-1,i} \;=\; b_{k,i} \cdot c_{k,i+1}, \\
c_{n,i} \cdot b_{n,i} = e^{z_{n,i+1}-z_{n,i}}, \qquad a_{n,n} \cdot b_{n,n} = e^{-2z_{n,n}}.
\end{gathered}
\tag{5.12}
$$

5.4 D_n Diagram

Diagram for D_n has the following form ($n = 4$)

$$
\begin{tikzcd}
 & & z_{33} \arrow[r, equal] \arrow[d, equal] & x_{44} & & & \\
 & z_{22} \arrow[r, equal] \arrow[d, equal] & x_{33} & z_{33} \arrow[l, "c_{33}"'] \arrow[u, "d_{33}"] & x_{43} \arrow[l, "a_{43}"'] & & \\
z_{11} \arrow[r, equal] \arrow[d, equal] & x_{22} & z_{22} \arrow[l, "c_{22}"'] \arrow[u, "d_{22}"] & x_{32} \arrow[l, "a_{32}"'] \arrow[u, "b_{32}"] & z_{32} \arrow[l, "c_{32}"'] \arrow[u, "d_{32}"] & x_{42} \arrow[l, "a_{42}"'] & \\
x_{11} & z_{11} \arrow[l, "c_{11}"'] \arrow[u, "d_{11}"] & x_{21} \arrow[l, "a_{21}"'] \arrow[u, "b_{21}"] & z_{21} \arrow[l, "c_{21}"'] \arrow[u, "d_{21}"] & x_{31} \arrow[l, "a_{31}"'] \arrow[u, "b_{31}"] & z_{31} \arrow[l, "c_{31}"'] \arrow[u, "d_{31}"'] & x_{41} \arrow[l, "a_{41}"']
\end{tikzcd}
\tag{5.13}
$$

Note that Lie algebra D_n can be realized as a fixed point subalgebra of A_{2n-1} using the involution

$$\iota: \quad X \longmapsto w_0^{-1} X^T w_0, \tag{5.14}$$

where w_0 is the longest element of Weyl group $\mathfrak{W}(A_{2n-1}) = \mathfrak{S}_{2n}$ and X^T denotes the standard transposition. Correspondingly the diagram for D_n can be obtain from that for A_{2n-1} by the identification of the variables assigned to the vertexes of A_{2n-1} diagram

$$w_0: \qquad x_{k,i} \longleftrightarrow -x_{k,k+1-i}. \tag{5.15}$$

An analog of the monomial relations (5.2) is as follows. Let us introduce new variables associate to the arrows of the diagram

$$
\begin{aligned}
&a_{l,j} = e^{z_{l-1,j}-x_{l,j}}, && 1 < l \leq n,\ 1 \leq j \leq l, \\
&b_{k,i} = e^{z_{k,i+1}-x_{k,i}} \ \ c_{k,i} = e^{x_{k,i}-z_{k,i}},\ \ d_{k,i} = e^{x_{k+1,i}-z_{k,i}}, && 1 \leq k < n,\ 1 \leq i \leq k.
\end{aligned}
\tag{5.16}
$$

The following defining relations hold

$$
\begin{aligned}
c_{k,i} \cdot b_{k,i} &= d_{k,i} \cdot a_{k+1,i+1}, \\
a_{k,i} \cdot d_{k-1,i} &= b_{k,i} \cdot c_{k,i+1}, \\
a_{k,i} \cdot d_{k-1,i} &= e^{x_{k,i+1}-x_{k,i}}.
\end{aligned}
\tag{5.17}
$$

Finally let us note that it is easy to check that the potentials $\mathcal{F}(x,z)$ obtained from B_n, C_n and D_n diagrams by summing the function assigned to the arrows coincide with that entering the integral representations in Section 4.

6 Elementary intertwiners for closed Toda chains

In this section we generalize the construction of the elementary intertwiners to the classical series of affine Lie algebras. For the necessary facts in the theory of affine Lie algebras

see [K], [DS]. Let us first recall the construction of the Q-operator for $A_n^{(1)}$ closed Toda chain [PG]. The integral kernel of the intertwining Q-operator in this case reads

$$Q^{A_n^{(1)}}(x_1,\dots,x_{n+1};\,y_1,\dots,y_{n+1}) = \exp\Big\{\sum_{i=1}^{n+1}(e^{x_i-y_i}+g_{i+1}e^{y_{i+1}-x_i})\Big\},\quad y_{n+2}=y_1, \tag{6.1}$$

The corresponding integral operator intertwines the following Hamiltonians operators for $A_n^{(1)}$ closed Toda chains

$$\mathcal{H}^{A_n^{(1)}}(x_i) = -\frac{1}{2}\sum_{i=1}^{n+1}\frac{\partial^2}{\partial x_i^2} + g_1e^{x_1-x_{n+1}} + \sum_{i=1}^{n} g_{i+1}e^{x_{i+1}-x_i}, \tag{6.2}$$

$$\mathcal{H}^{A_n^{(1)}}(y_i) = -\frac{1}{2}\sum_{i=1}^{n+1}\frac{\partial^2}{\partial y_i^2} + g_1e^{y_1-y_{n+1}} + \sum_{i=1}^{n} g_{i+1}e^{y_{i+1}-y_i}\,. \tag{6.3}$$

Below we provide kernels of the integral operators intertwining Hamiltonians of the closed Toda chains corresponding to other classical series of affine Lie algebras.

6.1 $A_{2n}^{(2)} \leftrightarrow BC_{n+1}^{(2)}$

Simple roots of the twisted affine root system $A_{2n}^{(2)}$ can be expressed in terms of the standard basis $\{e_i\}$ as follows

$$\alpha_1=e_1,\qquad \alpha_{i+1}=e_{i+1}-e_i,\qquad 1\le i\le n-1\qquad \alpha_{n+1}=-2e_n. \tag{6.4}$$

Corresponding Dynkin diagram is given by

$$\alpha_1 \Longleftarrow \alpha_2 \longleftarrow \dots \longleftarrow \alpha_{n-1} \Longleftarrow \alpha_n.$$

Simple roots of the twisted affine non-reduced root system $BC_n^{(2)}$ are given by

$$\alpha_0=2e_1,\quad \alpha_1=e_1,\quad \alpha_{i+1}=e_{i+1}-e_i,\ \ 1\le i\le n-1,\quad \alpha_{n+1}=-e_n-e_{n-1} \tag{6.5}$$

and the corresponding Dynkin diagram is as follows

$$\begin{array}{ccccccccc} \frac{\alpha_0}{\alpha_1} \Longleftrightarrow \alpha_2 & \longleftarrow & \dots & \longleftarrow & \alpha_{n-1} & \longleftarrow & \alpha_n \\ & & & & \uparrow & & \\ & & & & \alpha_{n+1} & & \end{array} \tag{6.6}$$

The integral operator with the following kernel

$$Q^{BC_{n+1}^{(2)}}_{A_{2n}^{(2)}}(x_i,\,z_i) = \exp\Big\{g_1e^{z_1}+\sum_{i=1}^{n}\Big(e^{x_i-z_i}+g_{i+1}e^{z_{i+1}-x_i}\Big)+g_{n+2}e^{-z_{n+1}-x_n}\Big\}, \tag{6.7}$$

intertwines Hamiltonian operators for $A^{(2)}_{2n}$ and $BC^{(2)}_{n+1}$

$$\mathcal{H}^{A^{(2)}_{2n}}(x_i) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + g_1 e^{x_1} + \sum_{i=1}^{n-1} g_{i+1}e^{x_{i+1}-x_i} + 2g_{n+1}g_{n+2}e^{-2x_n}, \tag{6.8}$$

$$\mathcal{H}^{BC^{(2)}_{n+1}}(z_i) = -\frac{1}{2}\sum_{i=1}^{n+1}\frac{\partial^2}{\partial z_i^2} + \frac{g_1}{2}\Big(e^{z_1} + g_1 e^{2z_1}\Big) + \sum_{i=1}^{n} g_{i+1}e^{z_{i+1}-z_i} + g_{n+2}e^{-z_{n+1}-z_n}. \tag{6.9}$$

The integral kernel for the inverse transformation is given by

$$Q_{\;BC^{(2)}_{n+1}}^{\;A^{(2)}_{2n}}(x_i,\, z_i) = Q_{\;A^{(2)}_{2n}}^{\;BC^{(2)}_{n+1}}(z_i,\, x_i). \tag{6.10}$$

6.2 $A^{(2)}_{2n-1} \leftrightarrow A^{(2)}_{2n-1}$

Simple roots of the twisted affine root system $A^{(2)}_{2n-1}$ are given by

$$\alpha_1 = 2e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad 1 \le i \le n-1 \qquad \alpha_{n+1} = -e_n - e_{n-1}, \tag{6.11}$$

and corresponding Dynkin diagram is

$$\begin{array}{ccccccccc} \alpha_1 & \Longrightarrow & \alpha_2 & \longrightarrow & \ldots & \longrightarrow & \alpha_{n-1} & \longrightarrow & \alpha_n \\ & & & & & & \big\downarrow & & \\ & & & & & & \alpha_{n+1} & & \end{array}$$

The integral operator represented by the following kernel

$$\begin{gathered} Q_{\;A^{(2)}_{2n-1}}^{\;A^{(2)}_{2n-1}}(x_i,\, z_i) = \\ = \exp\Big\{ g_1 e^{x_1+z_1} + \sum_{i=1}^{n-1}\Big(e^{x_i-z_i} + g_{i+1}e^{z_{i+1}-x_i}\Big) + e^{x_n-z_n} + g_{n+1}e^{-x_n-z_n}\Big\}, \end{gathered} \tag{6.12}$$

intertwines Hamiltonian operators for $A^{(2)}_{2n-1}$ closed Toda chains with different coupling constants

$$\mathcal{H}^{A^{(2)}_{2n-1}}(x_i) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + 2g_1 e^{2x_1} + \sum_{i=1}^{n-1} g_{i+1}e^{x_{i+1}-x_i} + g_n g_{n+1}e^{-x_n-x_{n-1}}, \tag{6.13}$$

$$\widetilde{\mathcal{H}}^{A^{(2)}_{2n-1}}(z_i) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial z_i^2} + g_1 g_2 e^{z_1+z_2} + \sum_{i=1}^{n-1} g_{i+1}e^{z_{i+1}-z_i} + 2g_{n+1}e^{-2z_n}. \tag{6.14}$$

6.3 $B_n^{(1)} \leftrightarrow BC_n^{(1)}$

Simple roots of the affine root system $B_n^{(1)}$ are given by

$$\alpha_1 = e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad 1 \le i \le n-1 \qquad \alpha_{n+1} = -e_n - e_{n-1}, \tag{6.15}$$

and corresponding Dynkin diagram is as follows

$$\begin{array}{ccccccc} \alpha_1 \Longleftarrow \alpha_2 & \longleftarrow & \ldots & \longleftarrow & \alpha_{n-1} & \longleftarrow & \alpha_n \\ & & & & \uparrow & & \\ & & & & \alpha_{n+1} & & \end{array}$$

Simple roots of the affine non-reduced root system $BC_n^{(1)}$ are

$$\alpha_0 = 2e_1, \quad \alpha_1 = e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad 1 \le i \le n-1 \qquad \alpha_{n+1} = -2e_n, \tag{6.16}$$

and corresponding Dynkin diagram is given by

$$\frac{\alpha_0}{\alpha_1} \Longleftrightarrow \alpha_2 \longleftarrow \ldots \longleftarrow \alpha_n \Longleftarrow \alpha_{n+1}$$

The integral operator represented by the following kernel

$$\begin{gathered} Q_{B_n^{(1)}}^{\;BC_n^{(1)}}(x_i,\, z_i) = \\ = \exp\left\{ g_1 e^{z_1} + \sum_{i=1}^{n-1} \left(e^{x_i - z_i} + g_{i+1} e^{z_{i+1} - x_i} \right) + e^{x_n - z_n} + g_{n+1} e^{-x_n - z_n} \right\}, \end{gathered} \tag{6.17}$$

intertwines Hamiltonians of $B_n^{(1)}$ and $BC_n^{(1)}$ closed Toda chains

$$\mathcal{H}^{B_n^{(1)}}(x_i) = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + g_1 e^{x_1} + \sum_{i=1}^{n-1} g_{i+1} e^{x_{i+1} - x_i} + g_n g_{n+1} e^{-x_n - x_{n-1}}, \tag{6.18}$$

$$\mathcal{H}^{BC_n^{(1)}}(z_i) = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2} + \frac{g_1}{2}\left(e^{z_1} + g_1 e^{2z_1} \right) + \sum_{i=1}^{n-1} g_{i+1} e^{z_{i+1} - z_i} + 2g_{n+1} e^{-2z_n}. \tag{6.19}$$

The integral kernel for the inverse transformation is given by

$$Q_{BC_n^{(1)}}^{\;B_n^{(1)}}(x_i,\, z_i) = Q_{B_n^{(1)}}^{\;BC_n^{(1)}}(z_i,\, x_i). \tag{6.20}$$

6.4 $C^{(1)} \leftrightarrow D^{(1)}$

Simple roots of the affine root system $C_n^{(1)}$ are

$$\alpha_1 = 2e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad 1 \le i \le n-1 \qquad \alpha_{n+1} = -2e_n, \tag{6.21}$$

and corresponding Dynkin diagram is given by

$$\alpha_1 \Longrightarrow \alpha_2 \longrightarrow \ldots \longleftarrow \alpha_n \Longleftarrow \alpha_{n+1}$$

Simple roots of the affine root system $D_n^{(1)}$ are

$$\alpha_1 = e_1 + e_2, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad 1 \le i \le n-1 \qquad \alpha_{n+1} = -e_n - e_{n-1}, \tag{6.22}$$

and corresponding Dynkin diagram is given by

$$\begin{array}{ccccccccc} \alpha_1 & \longleftarrow & \alpha_3 & \longleftarrow & \ldots & \longleftarrow & \alpha_{n-1} & \longleftarrow & \alpha_n \\ & & \Big\uparrow & & & & \Big\uparrow & & \\ & & \alpha_2 & & & & \alpha_{n+1} & & \end{array}$$

The integral operator with the following kernel

$$Q_{C_n^{(1)}}^{D_{n+1}^{(1)}}(x_i,\, z_i) = \exp\left\{ g_1 e^{x_1+z_1} + \sum_{i=1}^{n} \left(e^{x_i - z_i} + g_{i+1} e^{z_{i+1}-x_i} \right) + g_{n+2} e^{-z_{n+1}-x_n} \right\} \tag{6.23}$$

intertwines Hamiltonian operators for $C_n^{(1)}$ and $D_{n+1}^{(1)}$ closed Toda chains

$$\mathcal{H}^{C_n^{(1)}}(x_i) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + 2g_1 e^{2x_1} + \sum_{i=1}^{n-1} g_{i+1} e^{x_{i+1}-x_i} + 2g_{n+1}g_{n+2}e^{-2x_n}, \tag{6.24}$$

$$\mathcal{H}^{D_{n+1}^{(1)}}(z_i) = -\frac{1}{2}\sum_{i=1}^{n+1}\frac{\partial^2}{\partial z_i^2} + g_1 g_2 e^{z_1+z_2} + \sum_{i=1}^{n} g_{i+1} e^{z_{i+1}-z_i} + g_{n+2}e^{-z_{n+1}-z_n}. \tag{6.25}$$

The integral operator with the kernel

$$Q_{D_n^{(1)}}^{C_{n-1}^{(1)}}(x_i,\, z_i) = \exp\left\{ g_1 e^{x_1+z_1} + \sum_{i=1}^{n-1} \left(e^{z_i - x_i} + g_{i+1} e^{x_{i+1}-z_i} \right) + g_{n+1} e^{-x_n-z_{n-1}} \right\} \tag{6.26}$$

intertwines Hamiltonian operators for $D_n^{(1)}$ and $C_{n-1}^{(1)}$ closed Toda chains

$$\mathcal{H}^{D_n^{(1)}}(x_i) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + g_1 g_2 e^{x_1+x_2} + \sum_{i=1}^{n-1} g_{i+1} e^{x_{i+1}-z_i} + g_{n+1}e^{-x_n-x_{n-1}}, \tag{6.27}$$

$$\mathcal{H}^{C_{n-1}^{(1)}}(z_i) = -\frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial^2}{\partial z_i^2} + 2g_1 e^{2z_1} + \sum_{i=1}^{n-2} g_{i+1} e^{z_{i+1}-z_i} + 2g_n g_{n+1}e^{-2z_{n-1}}. \tag{6.28}$$

The integral kernel for the inverse transformation is given by

$$Q_{C_n^{(1)}}^{D_n^{(1)}}(x_i,\, z_i) = Q_{D_n^{(1)}}^{C_n^{(1)}}(z_i,\, x_i). \tag{6.29}$$

7 Baxter Q-operators

Now we apply the results presented in the previous Sections to the construction of the Baxter integral Q-operators for all classical series of affine Lie algebras. Let us note that the elementary intertwining operators and recursive operators for finite Lie algebras can be obtained from the elementary intertwining operators and Baxter operators for affine Lie algebras by taking appropriate limits $g_i \to 0$ in (6.7)-(6.26). This generalizes known relation between Baxter operator for $A_n^{(1)}$ and recursive operators for A_n.

The integral kernels for Q-operators have the following form

$$Q^{A_{2n}^{(2)}}(x_1,\ldots,x_n;\,y_1,\ldots,y_n) = \int \bigwedge_{i=1}^{n+1} dz_i Q_{A_{2n}^{(2)}}^{BC_{n+1}^{(2)}}(x_1,\ldots,x_n;\,z_1,\ldots,z_{n+1}) \times \tag{7.1}$$

$$\times Q_{BC_{n+1}^{(2)}}^{A_{2n}^{(2)}}(z_1,\ldots,z_{n+1};\,y_1,\ldots,y_n),$$

$$Q^{A_{2n-1}^{(2)}}(x_1,\ldots,x_n;\,y_1,\ldots,y_n) = \int \bigwedge_{i=1}^{n} dz_i\, Q_{A_{2n-1}^{(2)}}^{A_{2n-1}^{(2)}}(x_1,\ldots,x_n;\,z_1,\ldots,z_n) \times \tag{7.2}$$

$$\times Q_{A_{2n-1}^{(2)}}^{A_{2n-1}^{(2)}}(y_1,\ldots,y_n;\,z_1,\ldots,z_n),$$

$$Q^{B_n^{(1)}}(x_1,\ldots,x_n;\,y_1,\ldots,y_n) = \int \bigwedge_{i=1}^{n} dz_i\, Q_{B_n^{(1)}}^{BC_n^{(1)}}(x_1,\ldots,x_n;\,z_1,\ldots,z_n) \times \tag{7.3}$$

$$Q_{BC_n^{(1)}}^{B_n^{(1)}}(z_1,\ldots,z_n;\,y_1,\ldots,y_n),$$

$$Q^{C_n^{(1)}}(x_1,\ldots,x_n;\,y_1,\ldots,y_n) = \int \bigwedge_{i=1}^{n+1} dz_i\, Q_{C_n^{(1)}}^{D_{n+1}^{(1)}}(x_1,\ldots,x_n;\,z_1,\ldots,z_{n+1}) \times \tag{7.4}$$

$$Q_{D_{n+1}^{(1)}}^{C_n^{(1)}}(z_1,\ldots,z_{n+1};\,y_1,\ldots,y_n),$$

$$Q^{D_n^{(1)}}(x_1,\ldots,x_n;\,y_1,\ldots,y_n) = \int \bigwedge_{i=1}^{n-1} dz_i\, Q_{D_n^{(1)}}^{C_{n-1}^{(1)}}(x_1,\ldots,x_n;\,z_1,\ldots,z_{n-1}) \times \tag{7.5}$$

$$Q_{C_{n-1}^{(1)}}^{D_n^{(1)}}(z_1,\ldots,z_{n-1};\,y_1,\ldots,y_n).$$

8 Baxter operators for B_∞, C_∞ and D_∞

Similar approach can be applied to construct Baxter Q operators for infinite root systems B_∞, C_∞, BC_∞ and D_∞.

Simple roots and Dynkin diagrams for infinite Lie algebras A_∞, B_∞, C_∞, BC_∞ and D_∞ are as follows

$$A_\infty : \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad i \in \mathbb{Z}, \tag{8.1}$$

$$\ldots \longrightarrow \alpha_{-1} \longrightarrow \alpha_0 \longrightarrow \alpha_1 \longrightarrow \ldots \tag{8.2}$$

$$B_\infty : \quad \alpha_1 = e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad i \in \mathbb{Z}_{>0}, \tag{8.3}$$

$$\alpha_1 \Longleftarrow \alpha_2 \longleftarrow \alpha_3 \longleftarrow \ldots \tag{8.4}$$

$$C_\infty : \quad \alpha_1 = 2e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad i \in \mathbb{Z}_{>0}, \tag{8.5}$$

$$\alpha_1 \Longrightarrow \alpha_2 \longrightarrow \alpha_3 \longrightarrow \ldots \tag{8.6}$$

$$D_\infty : \quad \alpha_1 = e_1 + e_2, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad i \in \mathbb{Z}_{>0}, \tag{8.7}$$

$$\begin{array}{ccccccc} \alpha_1 & \longrightarrow & \alpha_3 & \longrightarrow & \alpha_4 & \longrightarrow & \ldots \\ & & \Big\uparrow & & & & \\ & & \alpha_2 & & & & \end{array} \tag{8.8}$$

$$BC_\infty : \quad \alpha_0 = 2e_1, \qquad \alpha_1 = e_1, \qquad \alpha_{i+1} = e_{i+1} - e_i, \qquad i \in \mathbb{Z}_{>0}, \tag{8.9}$$

$$\tfrac{\alpha_0}{\alpha_1} \Longleftrightarrow \alpha_2 \longleftarrow \alpha_3 \longleftarrow \ldots \tag{8.10}$$

Baxter Q-operator for A_∞ infinte Toda chain is known (see [T] for the classical limit). It is given by an integral operator with the kernel

$$Q(x_i,\, y_i) = \exp\Big\{ \sum_{i\in\mathbb{Z}} \big(e^{x_i - y_i} + g_i e^{y_{i+1} - x_i}\big) \Big\}, \tag{8.11}$$

and intertwines A_∞ Toda chain Hamiltonians

$$\mathcal{H}^{A_\infty}(x_i) = -\frac{1}{2}\sum_{i\in\mathbb{Z}} \frac{\partial^2}{\partial x_i^2} + \sum_{i\in\mathbb{Z}} g_i e^{x_{i+1} - x_i}, \tag{8.12}$$

$$\mathcal{H}^{A_\infty}(y_i) = -\frac{1}{2}\sum_{i\in\mathbb{Z}} \frac{\partial^2}{\partial y_i^2} + \sum_{i\in\mathbb{Z}} g_i e^{y_{i+1} - y_i}. \tag{8.13}$$

Its generalization to other classical series relies on the construction of the integral operators intertwining different classical series. Thus we have the following set of integral operators.

The integral operator with the kernel

$$Q_{B_\infty}^{\;BC_\infty}(x_i,\,z_i)=\exp\Big\{\,g_1e^{z_1}+\sum_{i>0}\big(e^{x_i-z_i}+g_{i+1}e^{z_{i+1}-x_i}\big)\,\Big\}, \tag{8.14}$$

intertwines B_∞ and BC_∞ Toda chain Hamiltonian operators

$$\mathcal{H}^{B_\infty}(x_i) \;=\; -\frac{1}{2}\sum_{i=1}^{\infty}\frac{\partial^2}{\partial x_i^2}+g_1e^{x_1}+\sum_{i=1}^{\infty}g_{i+1}e^{x_{i+1}-x_i}, \tag{8.15}$$

$$\mathcal{H}^{BC_\infty}(z_i) \;=\; -\frac{1}{2}\sum_{i=1}^{\infty}\frac{\partial^2}{\partial z_i^2}+\frac{g_1}{2}\Big(e^{z_1}+g_1e^{2z_1}\Big)+\sum_{i=1}^{\infty}g_{i+1}e^{z_{i+1}-z_i}. \tag{8.16}$$

Q-operator for B_∞ is then obtained by composition of the intertwiner operators. For the integral kernel of the Q-operator for B_∞ Toda chain we have

$$Q^{B_\infty}(x_i;y_i)=\int\bigwedge_{i=1}^{\infty}dz_i\,Q_{B_\infty}^{\;BC_\infty}(x_i,\,z_i)\cdot Q_{BC_\infty}^{\;B_\infty}(z_i,\,y_i).$$

Similarly the integral operator with the kernel

$$Q_{C_\infty}^{\;D_\infty}(x_i,\,z_i)=\exp\Big\{\,g_1e^{x_1+z_1}+\sum_{i>0}\big(e^{x_i-z_i}+g_{i+1}e^{z_{i+1}-x_i}\big)\,\Big\}, \tag{8.17}$$

intertwines C_∞ and D_∞ Toda chain Hamiltonian operators

$$\mathcal{H}^{C_\infty}(x_i) \;=\; -\frac{1}{2}\sum_{i=1}^{\infty}\frac{\partial^2}{\partial x_i^2}+2g_1e^{2x_1}+\sum_{i=1}^{\infty}g_{i+1}e^{x_{i+1}-x_i}, \tag{8.18}$$

$$\mathcal{H}^{D_\infty}(z_i) \;=\; -\frac{1}{2}\sum_{i=1}^{\infty}\frac{\partial^2}{\partial z_i^2}+g_1g_2e^{z_1+z_2}+\sum_{i=1}^{\infty}g_{i+1}e^{z_{i+1}-z_i}. \tag{8.19}$$

Thus integral Q-operator for C_∞ has as the kernel

$$Q^{C_\infty}(x_i;y_i)=\int\bigwedge_{i=1}^{\infty}dz_i\,Q_{C_\infty}^{\;D_\infty}(x_i,\,z_i)\cdot Q_{D_\infty}^{\;C_\infty}(z_i,\,y_i).$$

The integral operator with the kernel

$$Q_{D_\infty}^{\;C_\infty}(z_i,\,x_i)=\exp\Big\{\,g_1e^{x_1+z_1}+\sum_{i>0}\big(e^{z_i-x_i}+g_{i+1}e^{x_{i+1}-z_i}\big)\,\Big\}, \tag{8.20}$$

intertwines D_∞ and C_∞ Toda chain Hamiltonian operators

$$\mathcal{H}^{D_\infty}(x_i) \;=\; -\frac{1}{2}\sum_{i=1}^{\infty}\frac{\partial^2}{\partial x_i^2}+g_1g_2e^{x_1+x_2}+\sum_{i=1}^{\infty}g_{i+1}e^{x_{i+1}-x_i}, \tag{8.21}$$

$$\mathcal{H}^{C_\infty}(z_i) \;=\; -\frac{1}{2}\sum_{i=1}^{\infty}\frac{\partial^2}{\partial z_i^2}+2g_1e^{2z_1}+\sum_{i=1}^{\infty}g_{i+1}e^{z_{i+1}-z_i}. \tag{8.22}$$

Therefore Baxter integral Q-operator for D_∞ has the following kernel

$$Q^{D_\infty}(x_i; y_i) = \int \bigwedge_{i=1}^{\infty} dz_i \cdot Q_{D_\infty}^{C_\infty}(x_i,\, z_i) \cdot Q_{C_\infty}^{D_\infty}(z_i,\, y_i).$$

9 Conclusions

In this note we provide explicit expressions for recursive operators and Baxter Q-operators for classical series of Lie algebras. This allows us to generalize Givental representation for the eigenfunctions of the open Toda chain quadratic Hamiltonians to all classical series. In this note we consider only the case of zero eigenvalues leaving a rather straightforward generalization to non-zero eigenvalues to another occasion[GLO]. The proof of the eigenfunction property for the full set of Toda chain Hamiltonians follows the same strategy as in [Gi], [GKLO] and will be published separately. The results presented in this note provide a generalization to other classical series of only a part of [GKLO]. The other part connected with the interpretation of elementary intertwining operators in representation theory framework will discussed elsewhere.

Let us note that Q-operator was introduced by Baxter as a key tool to solve quantum integrable systems (see [B]). Therefore one can expect that the Givental representation and its generalizations should play an important role in the theory of quantum integrable systems solved by the quantum inverse scattering method (see e.g. [F]).

Finally let us mention two possible applications of the obtained results. The generalization of the Givental integral representation and corresponding diagrams to other classical series allows to describe flat torifications of the corresponding flag manifolds. This provides interesting applications to the explicit description of the mirror symmetry for flag manifolds. In particular one expects the mirror/Langlands duality between Gromov-Witten invariants for flag manifolds and the corresponding period integrals for B_n and C_n series. Another interesting application connected with the theory of automorphic forms and corresponding L-functions. Note that open Toda chain wave functions are given (see e.g. [STS]) by the Whittaker functions [Ha] for the corresponding Lie groups. Taking into account the simplicity and the unified form of the arising constructions of Whittaker functions for all classical groups one can expect important computational advantages in using the proposed integral representations.

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